An Approximate Analytical Solution of the Fractional Diffusion Equation with Absorbent Term and External Force by Homotopy Perturbation Method

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In the present paper, the approximate analytical solutions of a general diffusion equation with fractional time derivative in the presence of an absorbent term and a linear external force are obtained with the help of the powerful homotopy perturbation method (HPM). By using initial values, the approximate analytical solutions of the equation are derived. The results are deduced for different particular cases. The numerical results show that only a few iterations are needed to obtain accurate approximate solutions and these are presented graphically. The presented method is extremely simple, concise, and highly efficient as a mathematical tool in comparison with the other existing techniques.

Key words: Fractional Diffusion Equation, Fractional Brownion Motion, Homotopy Perturbation Method, Mittag-Leffler Function.

1. Introduction

The HPM is an approach for finding the approximate analytical solution of linear and nonlinear problems. The method was proposed by He [1, 2] and was successfully applied by him to solve nonlinear wave equation [3 – 7] and boundary value problems [8]. The method was used for integro differential equations by El-Shahed [9], non-Newtonian flow by Siddiqui et al. [10, 11], linear partial differential equations of fractional order by He [12], Momani and Odibat [13], Darvishi and Khani [14], Belendez et al. [15], Mousa and Ragab [16], etc. The fundamentals of the method can be found, for example, in He [17, 18]. The basic difference of this method from the other perturbation techniques is that it does not require small parameters in the equation which overcomes the limitations of traditional perturbation techniques.

We focus our attention to find the solution of the equation (Schot et al. [19])

$$\frac{\partial^\beta}{\partial t^\beta} u(x,t) = D \frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial x}[F(x)u(x,t)] \\ - \int_0^t \alpha(t-\xi)u(x,\xi)d\xi, \quad (1)$$

where D is a diffusion coefficient, $F(x)$ is an external force, $\alpha(t)$ is a time-dependent absorbent term which may be related to a reaction diffusion process. Additionally, we use the fractional Riemann-Liouville fractional integral operator of order $0 < \alpha \le 1$,

$$J_t^\alpha U(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{U(\xi)d\xi}{(t-\xi)^{1-\alpha}}, \quad t > 0.$$

The Caputo fractional derivative, applied to the time variable, is defined by

$$D_t^\alpha U(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{U^{(m)}(\xi)d\xi}{(t-\xi)^{\alpha+1-m}}, \\ m-1 < \alpha \le m, \, m \in N, \quad t > 0.$$

Here the external force (drift) and the absorbent rate (source) related to the reaction diffusion equation are physically interesting for their anomalous behaviour in aspect of Brownian motion. The presence of the reaction term used in (1) has been studied in catalytic processes in regular, heterogeneous or disorder systems, and also in solute transport through absorbent samples that are usually proportional to the concentration in the solution. Equation (1) may be related to the physical situations with fractional time derivatives of axisymmetric flow of a viscous fluid with turbulent diffusion, nonlinear diffusion in hard and soft superconductors, etc.

Though, the diffusion equations have been widely studied due to their various applications in physics and

engineering, but the study related to diffusion equations with nonlinear terms and fractional time derivatives are few in number (Bologna et al. [20], Lenzi et al. [21], etc.). Lenzi et al. [22] presented some classes of solutions of a general nonlinear fractional diffusion equation with absorbations. The similar study was made by Assis et al. [23]. Recently, Das [24] has used the variational iteration method to find the analytical solution of a fractional diffusion equation of order α ($0 < \alpha \leq 1$) only in the presence of external force. Schot et al. [19] has given an approximate solution of the equation with absorbent term and a linear external force in terms of Fox H-function [25]. Zahran [26] has given a closed form solution in Fox H-function of the generalized fractional reaction-diffusion equation subject to an external linear force field to describe the transport processes in disorder systems.

In this paper the homotopy perturbation method is used to solve the fractional diffusion equation problem in the presence of both linear external force and an absorbent term. Using the initial condition, the approximate analytical expressions of $u(x,t)$ for different Brownian motions are obtained. The effect of external force and absorbent term in the solution is obtained numerically for different particular cases, which are depicted graphically. The elegance of this method can be attributed to its simplistic approach in seeking the approximate analytical solution of the problem.

2. Fractional Diffusion Equation

We consider the fractional diffusion equation without any external force and in the presence of an absorbent term given by $\alpha(t) = \alpha t^{\beta-1}/\Gamma(\beta)$ (Schot et al. [19]), $\alpha > 0$, as

$$\frac{\partial^\beta}{\partial t^\beta} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\alpha}{s^\beta} \int_0^t (t-\xi)^{\beta-1} u(x,\xi) d\xi,$$
$$0 < \beta \leq 1. \tag{2}$$

Taking the Laplace transform of (2), we get

$$s^\beta \bar{u}(x,s) = \frac{d^2}{dx^2} \bar{u}(x,s) - \frac{\alpha}{s^\beta} \bar{u}(x,s), \tag{3}$$

where $\bar{u}(x,s) = L[u(x,t)]$.

Equation (3) can be written as

$$\frac{d^2}{dx^2} \bar{u}(x,s) - \left(\frac{\alpha}{s^\beta} + s^\beta \right) \bar{u}(x,s) = 0. \tag{4}$$

The solution of (4) is

$$\bar{u}(x,s) = \frac{1}{2} \sqrt{\frac{\alpha}{s^\beta} + s^\beta} \exp\left(-\sqrt{\frac{\alpha}{s^\beta} + s^\beta} |x| \right). \tag{5}$$

The whole hierarchies of moments $M_k(t) = <x^k(t)>$ (Giona and Roman [27], Mandelbrot and Wallis [28]) have the same time dependence as on fractional Brownian motion, which can be observed from the expressions

$$M_{2k}(t) = \frac{\Gamma(2k+1)\beta}{t} \sum_{r=0}^\infty \left[\frac{(-1)^r K_0{}^r t^{\beta(k+2r)}}{\Gamma(\beta(k+2r)+1)} \right], \tag{6}$$

$$M_{2k+1}(t) = M_{2k+1}^{\text{FBM}}(t) = 0, \tag{7}$$

where $K_0{}^r = [k(k+1)\dots(k+r-1)]\frac{(k+2r)}{\Gamma(r+1)}\alpha^r$.

To find the anomalous spreading of the displacement due to the presence of time fractional derivatives and a memory effect of the absorbent term, it is necessary to analyze the mean square displacement of a Brownian motion which is given by

$$\langle x^2(t) \rangle = 2\beta t^{\beta-1} E_{2\beta,\beta+1}(-K_1 t^{2\beta})$$
$$= 2\beta t^{\beta-1} \cos_{0,\beta}\left(\sqrt{K_1} t^\beta \right), \tag{8}$$

where $K_1{}^r = (2r+1)\alpha^r$, $E_{\lambda,\mu}(t) = \sum_{r=0}^\infty \frac{t^r}{(\lambda r+\mu)}$ is the Mittag-Leffler function in two parameters, and $\cos_{\lambda,\mu}(t) = \sum_{r=0}^\infty \frac{(-1)^r t^{2r}}{\Gamma(2\mu r+\mu-\lambda+1)}$ is the Cosine Mittag-Leffler function (Luchko and Srivastava [29]).

3. Solution of the Problem

Our aim is to solve the analytical fractional diffusion equation (1) for $D = 1$ and $F(x) = -kx$, i.e. the equation now becomes

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{\partial^2 u(x,t)}{\partial x^2} + k\frac{\partial}{\partial x}(x u(x,t))$$
$$- \int_0^t \alpha(t-\xi) u(x,\xi) d\xi \tag{9}$$

with the initial condition

$$u(x,0) = f(x). \tag{10}$$

Equation (9) can be written in operator form as

$$D_t{}^\beta u(x,t) = D_{xx} u(x,t) + kx D_x(u(x,t))$$
$$+ ku(x,t) - \int_0^t \alpha(t-\xi) u(x,\xi) d\xi, \tag{11}$$

where $D_t{}^\beta \equiv \frac{\partial^\beta}{\partial t^\beta}$.

According to the homotopy perturbation method, we construct the following homotopy

$$D_t{}^\beta u(x,t) = p[D_{xx}u(x,t) + kxD_x(u(x,t))$$
$$+ ku(x,t) - \int_0^t \alpha(t-\xi)u(x,\xi)d\xi], \quad (12)$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). In case $p = 0$, (12) becomes a linear equation, $D_t{}^\beta u = 0$, which is easy to solve [15, 30, 31]. Now applying the classical perturbation technique, we can assume that the solution of (9) can be expressed as a power series in p as

$$u(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t)$$
$$+ p^3u_3(x,t) + p^4u_4(x,t) + \dots. \quad (13)$$

When $p = 1$, (12) corresponds (11) and (13) becomes the approximate solution of (11), i.e. of (9). The convergence of the method has been proved in [2]. Substituting (13) in (12), and equating the terms with the identical powers of p, we can obtain a series of equations:

$$p^0 : D_t{}^\beta u_0(x,t) = 0, \quad (14)$$

$$p^1 : D_t{}^\beta u_1(x,t) = D_{xx}u_0(x,t) + kxD_x(u_0(x,t))$$
$$+ ku_0(x,t) - \int_0^t \alpha(t-\xi)u_0(x,\xi)d\xi, \quad (15)$$

$$p^2 : D_t{}^\beta u_2(x,t) = D_{xx}u_1(x,t) + kxD_x(u_1(x,t))$$
$$+ ku_1(x,t) - \int_0^t \alpha(t-\xi)u_1(x,\xi)d\xi, \quad (16)$$

$$p^3 : D_t{}^\beta u_3(x,t) = D_{xx}u_2(x,t) + kxD_x(u_2(x,t))$$
$$+ ku_2(x,t) - \int_0^t \alpha(t-\xi)u_2(x,\xi)d\xi, \quad (17)$$

$$p^4 : D_t{}^\beta u_4(x,t) = D_{xx}u_3(x,t) + kxD_x(u_3(x,t))$$
$$+ ku_3(x,t) - \int_0^t \alpha(t-\xi)u_3(x,\xi)d\xi, \quad (18)$$

and so on.

The method is based on applying the operator J_t^β (the inverse of Caputo operator $D_t{}^\beta$) on both sides of (14)−(18) to obtain the solutions of $u_i(x,t)$, $i \geq 0$, for different expressions of $\alpha(t)$.

Next we will discuss two particular examples for different types of absorbent terms which have physical importance.

Example 3.1. Taking $\alpha(t) = \frac{\alpha t^{\beta-1}}{\Gamma(\beta)}$, $0 < \beta \leq 1$, we obtain from (14)−(18),

$$u_0(x,t) = f(x), \quad (19)$$

$$u_1(x,t) = \phi_1(x)\frac{t^\beta}{\Gamma(\beta+1)} - \alpha f(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)}, \quad (20)$$

$$u_2(x,t) = \phi_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)} - 2\alpha\phi_1(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)}$$
$$+ \alpha^2 f(x)\frac{t^{4\beta}}{\Gamma(4\beta+1)}, \quad (21)$$

$$u_3(x,t) =$$
$$\phi_3(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)} - 3\alpha\phi_2(x)\frac{t^{4\beta}}{\Gamma(4\beta+1)}$$
$$+ 3\alpha^2\phi_1(x)\frac{t^{5\beta}}{\Gamma(5\beta+1)} - \alpha^3 f(x)\frac{t^{6\beta}}{\Gamma(6\beta+1)}, \quad (22)$$

$$u_4(x,t) = \phi_4(x)\frac{t^{4\beta}}{\Gamma(4\beta+1)}$$
$$- 4\alpha\phi_3(x)\frac{t^{5\beta}}{\Gamma(5\beta+1)} + 6\alpha^2\phi_2(x)\frac{t^{6\beta}}{\Gamma(6\beta+1)}$$
$$- 4\alpha^3\phi_1(x)\frac{t^{7\beta}}{\Gamma(7\beta+1)} + \alpha^4 f(x)\frac{t^{8\beta}}{\Gamma(8\beta+1)}, \quad (23)$$

where

$$\phi_1(x) = f''(x) + kxf'(x) + kf(x),$$
$$\phi_{r+1}(x) = \phi_r(x) + kx\phi_r(x) + k\phi_r(x),$$

and $f^{(r)}(x) = \frac{\partial^r}{\partial x^r}(f(x))$, $r \geq 1$.

Preceding in this manner the components u_n, $n \geq 0$, of the homotopy perturbation method can be completely obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution of $u(x,t)$ by the truncated series

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \quad (24)$$

where $\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$.

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [32].

Example 3.2. Taking $\alpha(t) = \frac{\alpha t^\sigma}{\Gamma(\sigma+1)}$, $0 < \sigma \leq 1$ (Zahran [26]), then from (14)−(18), we get

$$u_0(x,t) = f(x), \quad (25)$$

$$u_1(x,t) = \phi_1(x)\frac{t^\beta}{\Gamma(\beta+1)} - \alpha f(x)\frac{t^{\beta+\sigma+1}}{\Gamma(\beta+\sigma+2)}, \quad (26)$$

$$u_2(x,t) = \phi_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)}$$
$$- 2\alpha\phi_1(x)\frac{t^{2\beta+\sigma+1}}{\Gamma(2\beta+\sigma+2)}$$
$$+ \alpha^2 f(x)\frac{t^{2\beta+2\sigma+2}}{\Gamma(2\beta+2\sigma+3)}, \quad (27)$$

$$u_3(x,t) = \phi_3(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)}$$
$$- 3\alpha\phi_2(x)\frac{t^{3\beta+\sigma+1}}{\Gamma(3\beta+\sigma+2)}$$
$$+ 3\alpha^2\phi_1(x)\frac{t^{3\beta+2\sigma+2}}{\Gamma(3\beta+2\sigma+3)}$$
$$- \alpha^3 f(x)\frac{t^{3\beta+3\sigma+3}}{\Gamma(3\beta+3\sigma+4)}, \quad (28)$$

$$u_4(x,t) = \phi_4(x)\frac{t^{4\beta}}{\Gamma(4\beta+1)}$$
$$- 4\alpha\phi_3(x)\frac{t^{4\beta+\sigma+1}}{\Gamma(4\beta+\sigma+2)}$$
$$+ 6\alpha^2\phi_2(x)\frac{t^{4\beta+2\sigma+2}}{\Gamma(4\beta+2\sigma+3)}$$
$$- 4\alpha^3\phi_1(x)\frac{t^{4\beta+3\sigma+3}}{\Gamma(4\beta+3\sigma+4)}$$
$$+ \alpha^4 f(x)\frac{t^{4\beta+4\sigma+4}}{\Gamma(4\beta+4\sigma+5)}. \quad (29)$$

Proceeding in the similar manner the components u_n, $n \geq 0$, are obtained and finally the approximate analytical solution is obtained from (24).

4. Particular Cases

Case I: Here the expressions of displacement $u(x,t)$ are deduced for different particular cases for Example 3.1.

A. If $f(x) = x$, $\alpha = 0$, $k = 1$, i. e. in the presence of only the external force, the expression of the displacement becomes

$$u(x,t) = x\left[1 + \frac{2t^\beta}{\Gamma(\beta+1)} + \frac{4t^{2\beta}}{\Gamma(2\beta+1)}\right.$$
$$\left. + \frac{8t^{3\beta}}{\Gamma(3\beta+1)} + \frac{16t^{4\beta}}{\Gamma(4\beta+1)} + \cdots\right]$$

$$= x\sum_{r=0}^{\infty}\frac{2^r t^{r\beta}}{\Gamma(r\beta+1)} = xE_\beta(2t^\beta). \quad (30)$$

B. If $f(x) = x$, $\alpha = 1$, $k = 0$, i. e. in the presence of the absorbent term,

$$u(x,t) = x\left[1 - \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{4\beta}}{\Gamma(4\beta+1)}\right.$$
$$\left. - \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{t^{8\beta}}{\Gamma(8\beta+1)} - \cdots\right] \quad (31)$$

$$= x\sum_{r=0}^{\infty}\frac{(-1)^r t^{2r\beta}}{\Gamma(2r\beta+1)} = xE_{2\beta}(-t^{2\beta}).$$

C. If $f(x) = x$, $\alpha = 1$, $k = 1$, i. e. in the presence of both the linear external force and the absorbent term,

$$u(x,t) = x\left[1 + \frac{2t^\beta}{\Gamma(\beta+1)} + \frac{3t^{2\beta}}{\Gamma(2\beta+1)}\right.$$
$$\left. + \frac{4t^{3\beta}}{\Gamma(3\beta+1)} + \frac{5t^{4\beta}}{\Gamma(4\beta+1)} + \cdots\right] \quad (32)$$

$$= x\sum_{r=0}^{\infty}\frac{(r+1)t^{r\beta}}{\Gamma(r\beta+1)} = xE_\beta(K_2 t^\beta),$$

where $K_2{}^r = (r+1)$.

Case II: Here the expressions of $u(x,t)$ for $\sigma = \beta$ are obtained for the following particular cases for Example 3.2.

A. If $f(x) = x$, $\alpha = 0$, $k = 1$, i. e. in the presence of only the external force, the expression of the displacement becomes

$$u(x,t) = x\left[1 + \frac{2t^\beta}{\Gamma(\beta+1)} + \frac{4t^{2\beta}}{\Gamma(2\beta+1)}\right.$$
$$\left. + \frac{8t^{3\beta}}{\Gamma(3\beta+1)} + \frac{16t^{4\beta}}{\Gamma(4\beta+1)} + \cdots\right] \quad (33)$$

$$= x\sum_{r=0}^{\infty}\frac{2^r t^{r\beta}}{\Gamma(r\beta+1)} = xE_\beta(2t^\beta).$$

B. If $f(x) = x$, $\alpha = 1$, $k = 0$, i. e. in the presence of the absorbent term,

$$u(x,t) = x\left[1 - \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \frac{t^{2\beta+4}}{\Gamma(2\beta+5)}\right.$$
$$\left. - \frac{t^{3\beta+6}}{\Gamma(3\beta+7)} + \frac{t^{4\beta+8}}{\Gamma(4\beta+9)} - \cdots\right] \quad (34)$$

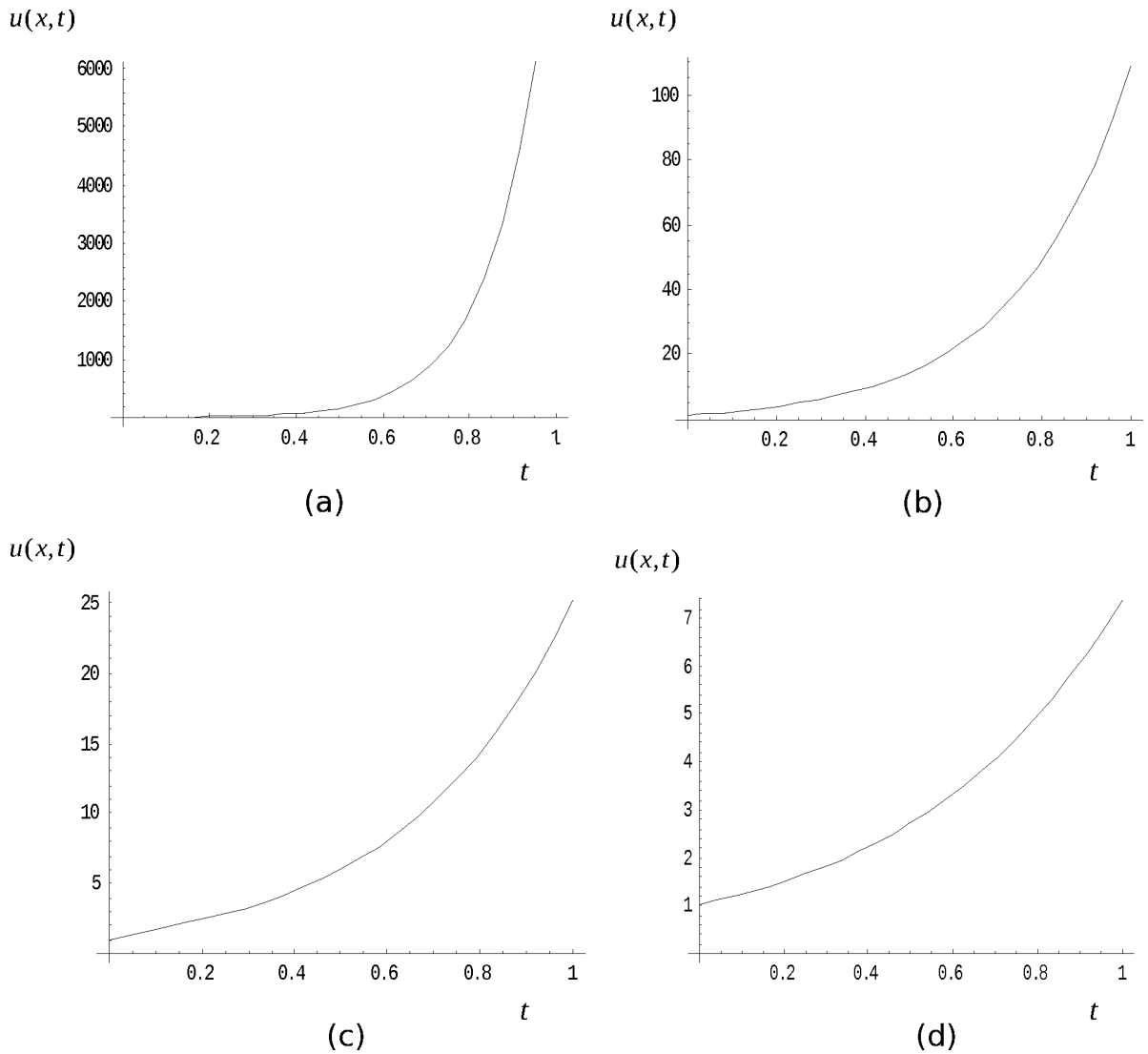$$= x\sum_{r=0}^{\infty}\frac{(-1)^r t^{r\beta+2}}{\Gamma(r(\beta+2)+1)} = xE_{\beta+2}(-t^{\beta+2}).$$

$u(x,t)$

(a)

$u(x,t)$

(b)

$u(x,t)$

(c)

$u(x,t)$

(d)

Fig. 1. (a) Plot of $u(x,t)$ vs. **t** at $x = \mathbf{1}$ for $\alpha = 0$, $k = 1$, $\beta = \frac{1}{3}$; (b) plot of $u(x,t)$ vs. **t** at $x = \mathbf{1}$ for $\alpha = 0$, $k = 1$, $\beta = \frac{1}{2}$; (c) plot of $u(x,t)$ vs. **t** at $x = \mathbf{1}$ for $\alpha = 0$, $k = 1$, $\beta = \frac{2}{3}$; (d) plot of $u(x,t)$ vs. **t** at $x = \mathbf{1}$ for $\alpha = 0$, $k = 1$, $\beta = 1$.

C. If $f(x) = x$, $\alpha = 1$, $k = 1$, i. e. in the presence of both the linear external force and the absorbent term,

$$
\begin{aligned}
u(x,t) = x\Bigg[& 1 + \frac{2t^{\beta}}{\Gamma(\beta+1)} - \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \frac{4t^{2\beta}}{\Gamma(2\beta+1)} - \frac{4t^{2\beta+2}}{\Gamma(2\beta+3)} + \frac{t^{2\beta+4}}{\Gamma(2\beta+5)} + \frac{8t^{3\beta}}{\Gamma(3\beta+1)} \\
& - \frac{12t^{3\beta+2}}{\Gamma(3\beta+3)} + \frac{6t^{3\beta+4}}{\Gamma(3\beta+5)} - \frac{t^{3\beta+6}}{\Gamma(3\beta+7)} + \frac{16t^{4\beta}}{\Gamma(4\beta+1)} - \frac{32t^{4\beta+2}}{\Gamma(4\beta+3)} + \frac{24t^{4\beta+4}}{\Gamma(4\beta+5)} \\
& - \frac{8t^{4\beta+6}}{\Gamma(4\beta+7)} + \frac{t^{4\beta+8}}{\Gamma(4\beta+9)} + \cdots \Bigg].
\end{aligned}
\tag{35}
$$

It is noted that only four terms of the series are used in evaluating the approximate analytical solution.
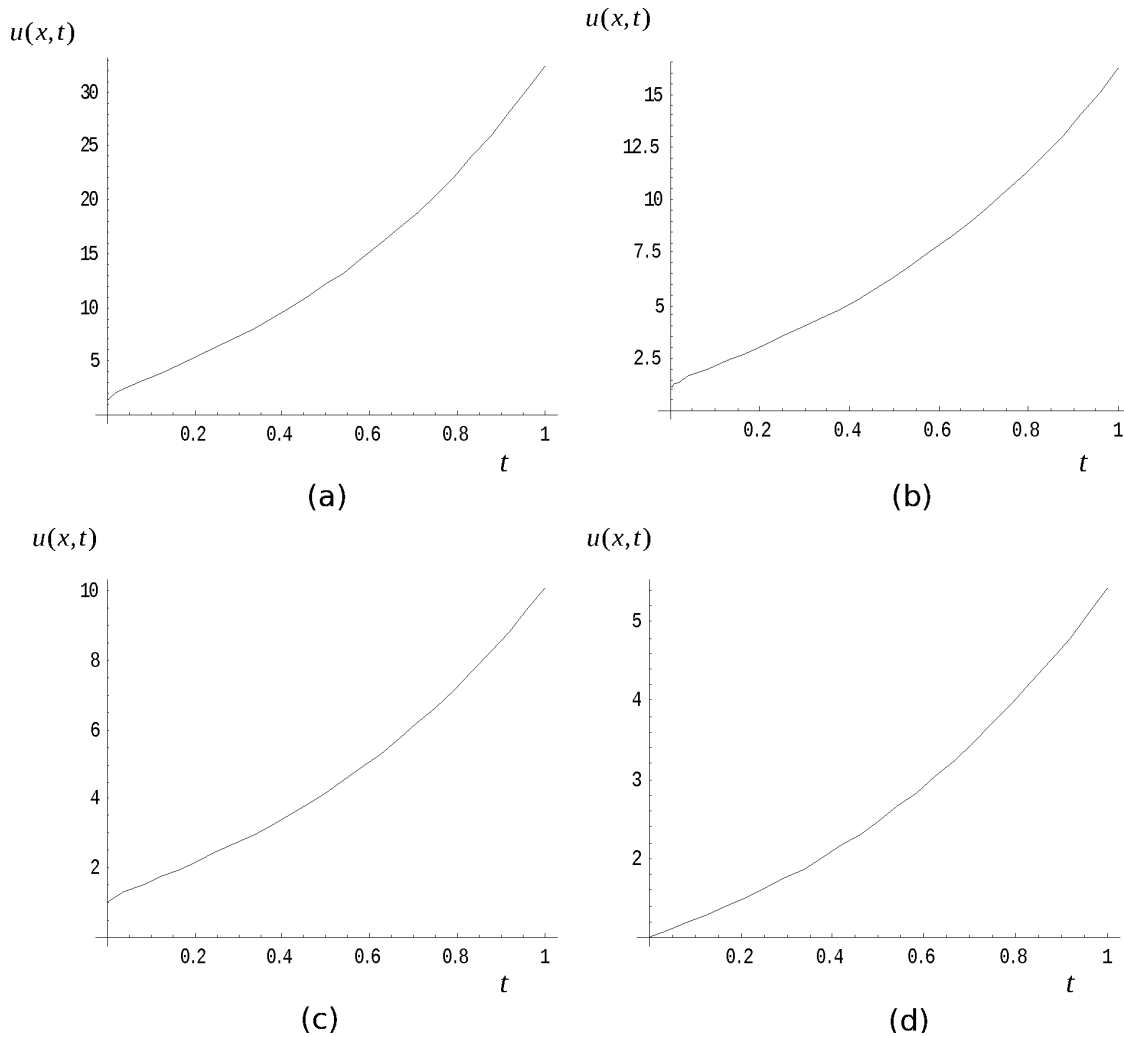
Fig. 2. (a) Plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 1$, $\beta = \frac{1}{3}$; (b) plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 1$, $\beta = \frac{1}{2}$; (c) plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 1$, $\beta = \frac{2}{3}$; (d) plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 1$, $\beta = 1$.

5. Numerical Results and Discussion

In this section, the numerical values of the displacement $u(x,t)$ for different fractional Brownian motions $\beta = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and also for the standard motion $\beta = 1$ are calculated for three different particular cases given in Case I and II, for various values of time t at $x = 1$. The results thus obtained are depicted through Figures $1-3$ for Case I and Figures $4-6$ for Case II.

It is seen from the Figure 1, i.e. for Case I-A that when there will be no absorbent term, i.e. when there exists only the external force, the values of $u(x,t)$ increase with the increase in t but rapidly decrease with the increase of β, which confirms

the exponential decay of regular Brownian motion. This result is in complete agreement with Giona and Roman [33].

The similar nature is seen in Figure 2 (Case I-B), where both the external force and the absorbent term exist. Here the magnitude of the displacement is low in comparison with the previous one. But it is observed from Figure 3 (Case I-C), that in the absence of the external force, i.e. only in presence of the absorbent term, the behaviour of the displacement is quite opposite in nature. Here the displacement decreases with the increase in t and slowly increases with increase in β. This confirms the anomalous distribution of the displacement with the initial condition connected to the
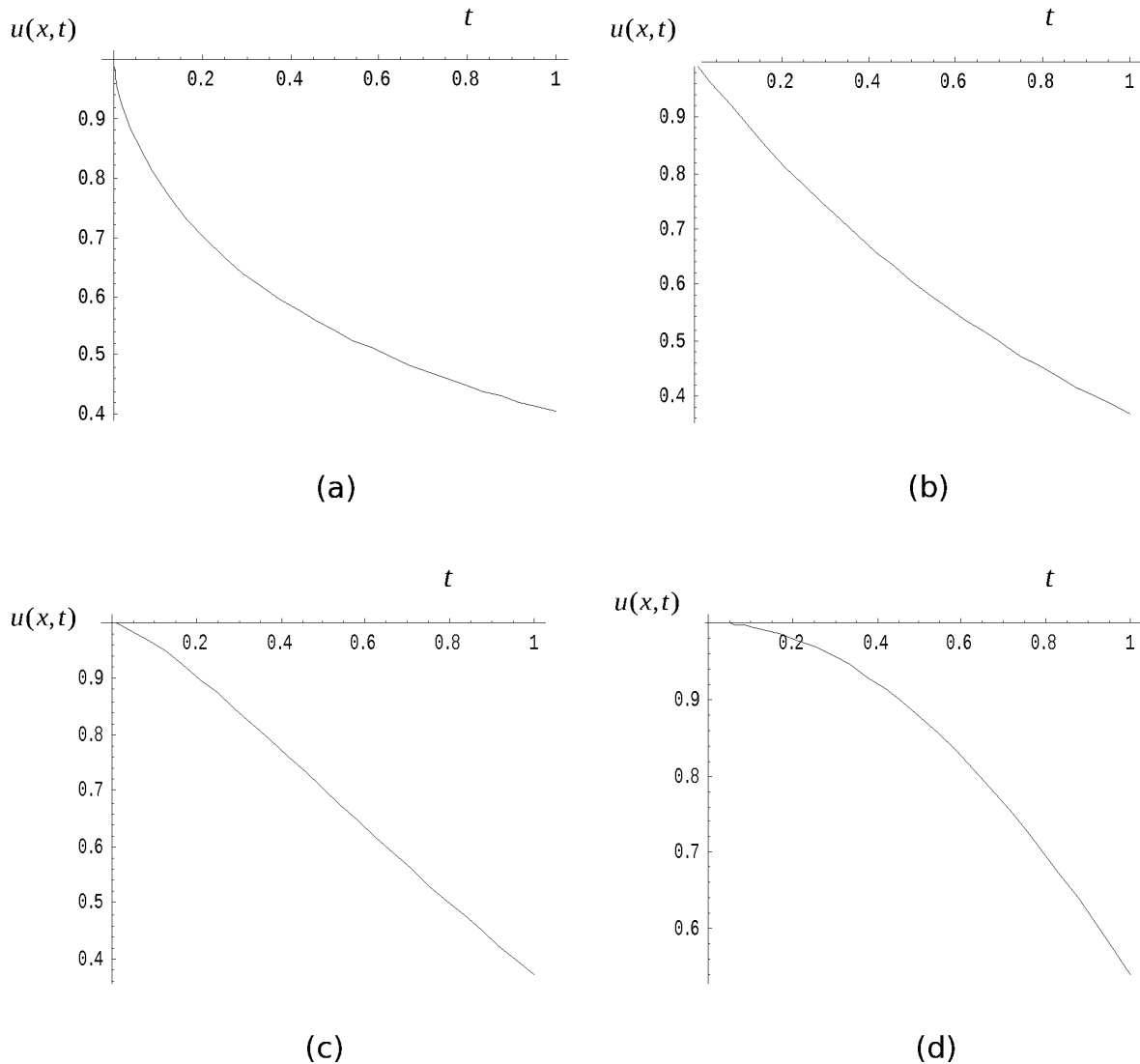
(a)

(b)

(c)

(d)

Fig. 3. (a) Plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 0$, $\beta = \frac{1}{3}$; (b) plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 0$, $\beta = \frac{1}{2}$; (c) plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 0$, $\beta = \frac{2}{3}$; (d) plot of $u(x,t)$ vs. t at $x = 1$ for $\alpha = 1$, $k = 0$, $\beta = 1$.

fractional time derivative due to the explicit effect of the absorbent term (Schot et. al. [19]).

It is noted from the Figures 1 – 3 that the absorbent term has a tremendous effect. In the absence of this term, the displacement tends to increase as the external force increases driving the system to the verge of instability. The dynamic response and stability margin can be improved with a damping force which is provided by the absorbent term.

In the absence of the external force, the displacement will be minimized by the action of the absorbent

term (Fig. 3). If the computations are carried for further time, the displacement finally goes to zero.

The nature of the Figures 4 and 5, which graphically represent Cases II-A and II-B, respectively, are similar to Figures 1 and 2. It is observed from Figure 4 that the increment of the displacement for Case II-A is lesser in comparison with Case I-A, in the presence of only external force. It is seen from Figure 5 that the increase of displacement due to the forced term has very less effect due to the source term. But Figure 6 which graphically describes Case II-C is completely opposite
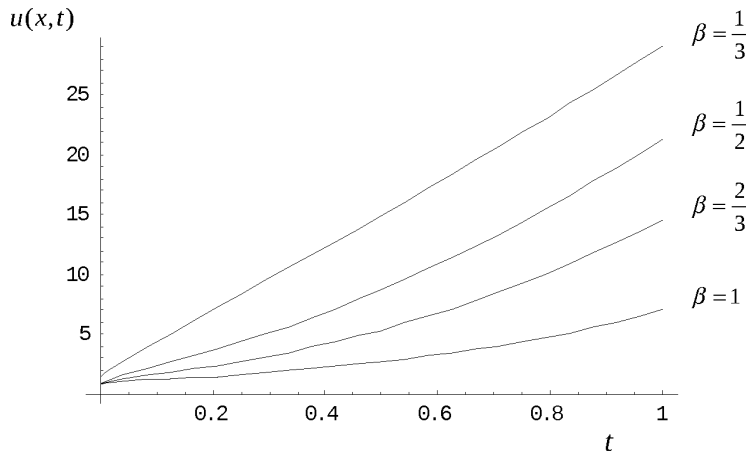
Fig. 4. Plot of $u(x,t)$ vs. t for various values of β at $x = 1$, $\alpha = 0$, $k = 1$.
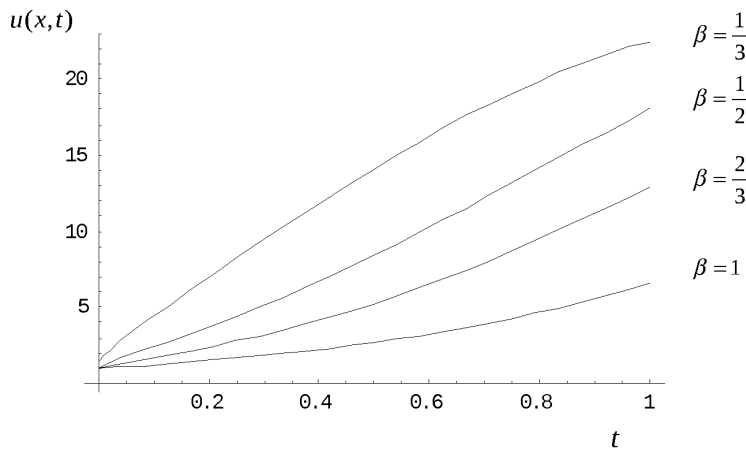

Fig. 5. Plot of $u(x,t)$ vs. t for various values of β at $x = 1$, $\alpha = 1$, $k = 1$.

in nature in comparison with Figure 3 for different values of fractional time derivatives. Here the displacement decreases with time and slowly decreases with the increase in β. This is due to the different form of the absorbent term.

6. Conclusion

There are two important goals that we have achieved for this study. The first one to employ the powerful HPM to investigate the general diffusion equation for different particular situations. HPM is a powerful mathematical tool which reduces the nonlinear problems to a set of ordinary differential equations to get the approximate analytical solution easily. Moreover it does not require small parameters in the equations which overcome the limitations of traditional perturbation techniques. This method is very effective, convenient, and supplies quantitatively reliable results.

Another important point of this study is to show the effect of external force, source term and also both the terms simultaneously on the fractional diffusion equation. Here it is seen that the absorbent term does not have any influence on the forced response on the system for the Example 3.2, whereas for Example 3.1, the absorbent term reduces the influence of the forcing term as reflected in the decrease of displacement term with time. So we can conclude that for Example 3.1 the absorbent term has a damping/stabilizing effect on the forced response of the system, whereas in Example 3.2 it does not have any such effect. The authors strongly believe the stability analysis discussed in this article in presence of these terms of physical interest will provide significant change from the usual approach to engineers and physicists working in this area of research.
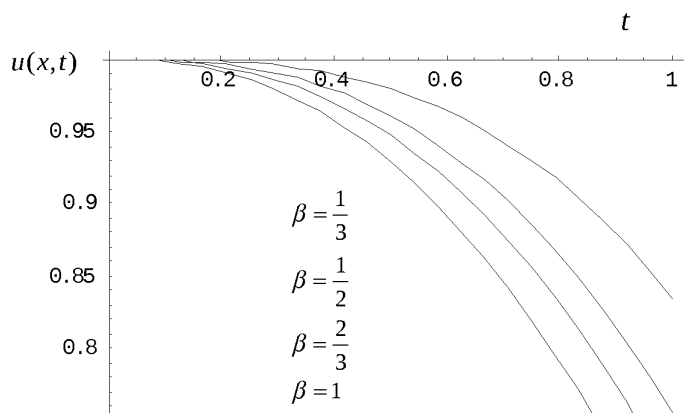
Fig. 6. Plot of $u(x,t)$ vs. t for various values of β at $x = 1$, $\alpha = 1$, $k = 0$.

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